ON THE TOPOLOGY OF THREE-DIMENSIONAL STEADY FLOWS OF AN IDEAL FLUID

(O TOPOLOGII TREKHMERNYKH STATSIONARNYKH TECHENII IDEAL'NOI ZHIDKOSTI)

PMM Vol. 30, No. 1, 1966, pp. 183-185

V.I. ARNOL'D
(Moscow)

(Received August 16, 1965)

We shall consider the rotational steady flows of an incompressible inviscid fluid in a bounded region *D*. It will be assumed that the vectors of velocity and vorticity are not everywhere colinear. It will be shown that the region of flow *D* is divided by the critical 'Bernoulli surfaces' into a finite number of cells, in each of which the streamlines are either closed, or else, everywhere they closely encircle toroidal surfaces.

1. The equations of motion. The Euler-Newton equation

$$\frac{d\mathbf{v}}{dt} = -\operatorname{grad} p, \quad \operatorname{div} v = 0 \qquad \left(\frac{dv}{dt} = \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} v\right) \tag{1.1}$$

is equivalent to the 'Bernoulli equation'

$$\frac{\partial \mathbf{v}}{\partial t} = [\mathbf{v}, \operatorname{curl} \mathbf{v}] - \operatorname{grad} \alpha, \quad \operatorname{div} v = 0 \qquad (\alpha = p + {}^1\!/_2 v^2) \tag{1.2}$$

For steady flow the Bernoulli equation takes the form

$$[v, \operatorname{curl} v] = \operatorname{grad} \alpha, \quad \operatorname{div} v = 0 \tag{1.3}$$

Let us make use of the well known identity of vector analysis

$$\operatorname{curl} [a, b] = \{b, a\} + a \operatorname{div} b - b \operatorname{div} a \tag{1.4}$$

Here $\{b, a\}$ is Poisson's bracket

$$\{b, a\}_i = \sum \frac{\partial a_i}{\partial x_j} b_j - \frac{\partial b_i}{\partial x_j} a_j$$

From the formulas (1.3) and (1.4) it follows that the velocity field of a steady flow commutates with its vorticity:

$$\{v, \operatorname{curl} v\} = 0 \tag{1.5}$$

We shall assume that the region of flow *D* is connected, finite and bounded by an analytical surface Γ; the boundary conditions are $(v, n)_{\Gamma} = 0$ (tangency).

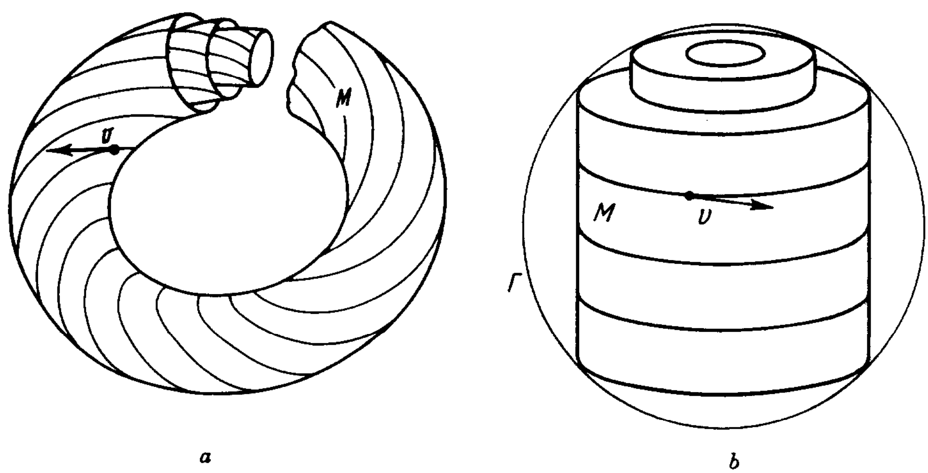
2. **Theorem**. Let v be an analytic, steady velocity field, not everywhere colinear with its vorticity

$$[v, \text{curl } v] \not\equiv 0 \tag{2.1}$$

Then, almost all the streamlines are either closed or everywhere dense on two-dimensional toruses: all streamlines of other type fill a finite number of analytic sub-manifolds of D.

Note. To remove the condition (2.1) is probably impossible, since flows with curl $v \equiv \lambda v$ (λ = const) can evidently have streamlines with very complex topology, typical for the problems of celestial mechanics (see [1], fig. 6). Such intricate streamlines, however, can also exist in steady flows of a viscous fluid, closely resembling the flows of an ideal fluid. We notice, moreover, that formulas (1.1) to (1.5) and the theorem together with its proof are easily applicable to the case of flow of an ideal fluid in three-dimensional Riemann space (see [2]).

3. **Proof**. Let us consider the level surfaces of the function α (see (1.3)). The connected components of these surfaces will be called Bernoulli surfaces. The streamlines and lines of vorticity, according to (1.3), are orthogonal to grad α and therefore lie on the Bernoulli surfaces. We shall show that the majority of Bernoulli surfaces are toruses or rings.



a b

FIG. 1 *a*, *b*

We shall call the value α_0 *poor* if there exists a point x in the region D where grad $\alpha = 0$ and $\alpha(x) = \alpha_0$, or, if there exists a point x on the boundary Γ, at which grad α is orthogonal to Γ and $\alpha(x) = \alpha_0$. From the analycity of α and Γ it follows that poor values of α are finite in number. The points x at which the function α takes poor values form a finite number of analytic sub-manifolds of D of dimensionality not higher than 2 (since the function α is not constant, see (2.1)). These sub-manifolds can be called poor, whilst all the remaining Bernoulli surfaces are good.

The poor sub-manifolds divide the region D into cells, each of which is stratified by good Bernoulli surfaces. A good Bernoulli surface, not intersecting with the boundary of the region Γ, is a closed smooth two-dimensional surface, since grad $\alpha \neq 0$ on it. It turns out that this surface is a torus (see the case (1) and fig. 1*a*).

A good Bernoulli surface intersecting with the boundary of the region Γ intersects with it transversely (since on the boundary grad α is not orthogonal to Γ). Therefore such a surface is smooth, with a boundary consisting of a finite number of smooth closed curves lying on Γ. It turns out that this surface is a ring (see case (2) and fig. 1*b*).

Case (1). Let M be on unbounded Bernoulli surface. Let us construct on M a system of angular co-ordinates α, β (mod 2π) so that the streamlines would have the equation $d\alpha / d\beta = \lambda = \text{const}$. This proves that M is a torus. But on the torus the lines $d\alpha / d\beta = \lambda$ are closed if λ is a rational number, and everywhere dense if λ is irrational. Therefore the theorem in case (1) is fully proved if the co-ordinate α, β can be constructed.

Let us consider a system of ordinary differential equations in $y\ (\tau,\ x,\ \sigma)$

$$\frac{dy}{d\tau} = s \operatorname{curl} v\,(y) + tv,\ \ y\,(0,\ x,\ \sigma) = x,\ \ \sigma = (s,\ t)$$

Here the parameter x is a point on the Bernoulli surface M, whilst σ is a point on the s, t-plane. Since the vectors v and curl v touch M, the point y lies on the same Bernoulli surface as x. When x is fixed, the formula

$$p_x(\sigma) = y\ (1,\ x,\ \sigma) \tag{3.1}$$

determines the mapping of the σ plane onto the Bernoulli surface M. From (1.5) the relation of commutativity follows

$$p_{p_{x(\sigma)}}\ (\sigma') = p_x\ (\sigma + \sigma') = p_{p_{x(\sigma')}}\ (\sigma) \tag{3.2}$$

Since the vectors v and curl v are linearly independent on M, the mapping (3.2) has a good overlap (i.e. the local value of σ can be taken as a co-ordinate on M). In fact, however, there are many points σ overlapping x. These points form, according to (3.2), a 'lattice' (if $p_x\ (\sigma) = p_x\ (\sigma') = x$, then also $p_x\ (\sigma + \sigma') = x$). From the commutativity of the Bernoulli surface M it follows, that this lattice has two generators σ_1 and σ_2 (two points of the plane such that any σ overlapping the point x has the form $m\sigma_1 + n\sigma_2$ with integral m and n). Let us make on the plane σ a linear substitution of the variables s and t by α and β, so that the co-ordinates of the points σ_1 and σ_2 would be $(2\pi,\ 0)$ and $(0,\ 2\pi)$. It is easy to see that α, β (mod 2π) are the required angular co-ordinates on the Bernoulli surface M. Hence the theorem is proved for the case (1).

Case (2). Let M be a Bernoulli surface with a boundary. The boundary of M consists of several closed streamlines lying on the boundary surface Γ (since the vector v is tangent to both M and Γ). Let x be a point on the boundary of M. Then, in the notation of (3.1), the closed streamline passing through x is

---

The above hypothesis was verified by M. Hennon by numerical experiment on the machine of the astrophysics institute in Paris.

$$p_x\ (0,\ t) = p_x\ (0,\ t+T) \quad (\infty < t < +\infty) \tag{3.3}$$

Let us put $z = p_x\ (s,\ t)$. Then, from the relations (3.2) and (3.3) it follows that

$$p_z\ (0,\ T) = p_x\ (s,\ t+T) = p_{p_x(0,T)}\ (s,\ t) = p_x\ (s,\ t) = z \tag{3.4}$$

i.e. the streamline passing through z is closed. But every point of M has the form $z = p_x\ (s,\ t)$ (in view of the linear independence of v and curl v and the connectedness of M). Therefore the formula (3.4) proves that all streamlines on M are closed. At the same time this formula introduces on M, the co-ordinates of the ring

$$t\ (\mathrm{mod}\ T),\ s, \quad 0 \leqslant s \leqslant S \ \text{ or } \ S \leqslant s \leqslant 0$$

Thus the proof of the theorem is completed.

BIBLIOGRAPHY

1. Arnol'd V.I. Malye znamenateli i problemy ustoichivosti dvizheniia v klassicheskoi i nebesnoi mekhanike (Small denominators and problems of stability of motion in classical and celestial mechanics). Uspekhi matem. nauk, Vol. 18, No. 6, pp 91-192, 1963.

2. Arnol'd V.I. Sur la topologie des écoulements stationaires des fluides parfaits. (Comptes Rendus Acad. Sci. Paris, Vol. 261, No. 1, 1965.

Translated by A.N.A.